

# A UNIVERSAL HYPERCYCLIC REPRESENTATION

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**ABSTRACT.** For any countable group, and also for any locally compact second countable, compactly generated topological group,  $G$ , we show the existence of a “universal” hypercyclic (i.e. topologically transitive) representation on a Hilbert space, in the sense that it simultaneously models every possible ergodic probability measure preserving free action of  $G$ .

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## 1. INTRODUCTION

A bounded linear operator  $S$  on a Banach space  $B$  is said to be *hypercyclic* if there are vectors  $v \in B$  such that the sequence  $\{S^n v\}_{n \geq 0}$  is dense in  $B$ . It is called *frequently hypercyclic* if there are vectors  $v$  such that for any non-empty open set  $U \subset B$  one has

$$(1.1) \quad \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mathbf{1}_U(S^n v) > 0.$$

One way to get such a property is for there to be a globally supported  $S$ -invariant probability measure  $\mu$  on  $B$  such that the dynamical system  $(B, S, \mu)$  is ergodic. In that case (1.1) will hold for  $\mu$ -a.e.  $v$  by Birkhoff’s ergodic theorem. We shall call  $S$  *universal* if for every ergodic probability measure preserving dynamical system  $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ , there exists an  $S$ -invariant probability measure  $\nu$  on  $B$  which is positive on every nonempty open subset of  $B$  and such that the dynamical systems  $\mathbf{X}$  and  $(B, \text{Borel}(B), \nu, S)$  are isomorphic. More generally, we have the following definition.

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**Definition 1.1.** For a topological group  $G$ , a linear representation as operators on a Banach space  $B$  will be called *universal* if for every ergodic probability measure preserving free  $G$ -action  $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$ , there exists an  $\{S_g\}_{g \in G}$ -invariant probability measure  $\nu$  on  $B$  which is positive on every nonempty open subset of  $B$  and such that the  $G$ -actions  $\mathbf{X}$  and  $(B, \text{Borel}(B), \nu, \{S_g\}_{g \in G})$  are isomorphic.

In this work we show the existence of a universal representation on Hilbert space for the following classes of groups.

1. All countable discrete groups.
2. All locally compact, second countable, compactly generated groups.
3. Groups  $G$  of the form  $G = \cup_{n=1}^{\infty} K_n$  where  $K_1 < K_2 < \dots$  is an increasing sequence of compact open subgroups.

The precise statement for a countable infinite group is as follows:

**Theorem 1.2.** *Let  $G$  be a countable infinite discrete group. There exists a faithful representation of  $G$ ,  $g \mapsto S_g$ , as a group of bounded linear operators on a separable Hilbert space  $H$  which is universal.*

Note that when a transformation  $T_g$  acts ergodically on  $(X, \mathcal{B}, \mu)$  it follows that the operator  $S_g$  is a frequently hypercyclic operator [1, Proposition 6.23]. In particular by choosing the  $G$ -system  $\mathbf{X}$  to be mixing (which is always possible), we see that each  $S_g$  with  $\{g^n : n \in \mathbb{Z}\}$  infinite, is frequently hypercyclic. In the special case where  $G = \mathbb{Z}$ , the group of integers, we obtain (as  $S_1$ ) a universal hypercyclic operator on Hilbert space.

We note that N. S. Feldman has previously proved the following topological version of our result for a hypercyclic operator ([3] and [1, Theorem 2.27]).

**Theorem 1.3.** *There exists a hypercyclic operator  $T$  acting on a separable Hilbert space  $\mathcal{H}$  which has the following property. For any compact metrizable space  $K$  and any continuous map  $f : K \rightarrow K$ , there exists a  $T$ -invariant compact set  $L \subset \mathcal{H}$  such that  $f$  and  $T \upharpoonright_L$  are topologically conjugate.*

In this theorem the operator is not invertible, it is in fact one of the simplest hypercyclic operators defined as follows. Let  $\mathcal{H}$  be the Hilbert space of sequences  $u = (u_n)_1^{\infty}$ , where  $u_n \in l_2(\mathbb{N})$  and  $\|u\|^2 = \sum_1^{\infty} \|u_n\|^2$ . The operator  $T$  is given by  $(Tu)_n = 2u_{n+1}$ . Since  $l_2(\mathbb{N})$  contains a copy of the Hilbert cube and any compact metric space  $K$  imbeds in the Hilbert cube, there is a homeomorphism  $\phi_0 : K \rightarrow l_2(\mathbb{N})$ . The map  $\phi : K \rightarrow \mathcal{H}$  is defined by

$$\phi(z)_n = \frac{1}{2^n} \phi_0(f^n z),$$

and it is easily seen that  $\phi$  intertwines the actions of  $f$  and  $T$ . Our operator will also be defined by first defining a function and then evaluating the function along orbits, but both the construction of the Hilbert space and the function are considerably more involved.

In Section 2 we handle the case where  $G$  is finitely generated and then, in Section 3, use an embedding theorem of Higman, Neumann and Neumann [4] to prove the theorem for general countable groups. In Section 4 we consider an analogous theorem for compactly generated, second countable locally compact groups.

Our doubts as to the existence of a topological analogue of the Higman-Neumann-Neumann theorem, which will ensure the possibility of embedding a general second countable locally compact topological group in a compactly generated group, were recently confirmed in [2]. Thus, if an extension of Theorem 4.2 to the general second countable locally compact group is valid, a different route needs to be found.

Finally, in the last section we will show that the theorem still holds for groups of the form  $G = \bigcup_{n=1}^{\infty} K_n$  where  $K_1 < K_2 < \dots$  is an increasing sequence of compact open subgroups.

In the proof of Theorem 1.2 it is not hard to see that our arguments work for any  $\ell_p(G, w)$  for  $1 < p < \infty$  (though not for  $\ell_1(G, w)$  as we need the convergence a.e. supplied by Theorem 2.3). We conclude the introduction with the following general problem.

**Problem 1.4.** Which separable Banach spaces (or more generally separable Fréchet spaces) can serve as universal models for ergodic, countable group actions, replacing Hilbert space in Theorem 1.2?

## 2. THE FINITELY GENERATED CASE

**2.1. The Hilbert space representation.** Let  $G$  be a finitely generated group with  $\mathfrak{a} = \{a_1^{\pm 1}, a_2^{\pm 1}, \dots, a_d^{\pm 1}\}$ , a symmetric set of generators. Set

$$(2.1) \quad \rho = \frac{1}{2d+1} \left( \delta_e + \sum_{i=1}^d \delta_{a_i^{\pm 1}} \right),$$

a probability measure on  $G$ . Let  $p_1, p_2, \dots$  be a sequence of positive real numbers with  $\sum_{n=1}^{\infty} p_n = 1$  and such that for some  $C > 0$ ,  $\frac{p_n}{p_{n+1}} \leq C$  for every  $n$ . We set

$$w(g) = \sum_{n=1}^{\infty} p_n \rho^{*n}(g) \quad g \in G.$$

Here  $\rho^{*n} = \rho * \rho * \dots * \rho$  ( $n$ -times) is the  $n$ -th convolutional power of  $\rho$ .

For  $\xi \in \mathbb{R}^G$  let

$$(2.2) \quad \|\xi\|^2 := \sum_{g \in G} \xi^2(g) w(g) = \sum_{n=1}^{\infty} \sum_{g \in G} p_n \xi^2(g) \rho^{*n}(g).$$

The Hilbert space  $H = l_2(G, w)$  is defined as:

$$\{\xi \in \mathbb{R}^G : \|\xi\| < \infty\}.$$

**Lemma 2.1.** For each  $a \in \mathfrak{a}$  the operator  $S_a : l_2(G, w) \rightarrow l_2(G, w)$  defined by

$$(S_a \xi)(g) = \xi(ga)$$

is a bounded linear operator with  $\|S_a\| \leq \sqrt{(2d+1)C}$  and the map  $a \mapsto S_a$  defines a linear representation of  $G$  in  $l_2(G, w)$ .

*Proof.* We write  $\sum_{g \in G} \xi^2(g) \rho^{*n}(g) = \mathbb{E} \xi^2(X_1 X_2 \dots X_n)$ , where  $X_1, X_2, \dots$  is an i.i.d. sequence of random variables with distribution  $\rho$ . Note that

$$\mathbb{E} \xi^2(X_1 X_2 \dots X_n a) \leq (2d+1) \mathbb{E} \xi^2(X_1 X_2 \dots X_n X_{n+1}),$$

hence by (2.2),

$$\begin{aligned}
\|S_a \xi\|^2 &= \sum_{g \in G} (S_a \xi)^2(g) w(g) \\
&= \sum_{n=1}^{\infty} \sum_{g \in G} p_n (S_a \xi)^2(g) \rho^{*n}(g) \\
&= \sum_{n=1}^{\infty} p_n \mathbb{E} \xi^2(X_1 X_2 \cdots X_n a) \\
&\leq (2d+1) \sum_{n=1}^{\infty} p_n \mathbb{E} \xi^2(X_1 X_2 \cdots X_n X_{n+1}) \\
&\leq (2d+1) \sum_{n=1}^{\infty} \frac{p_n}{p_{n+1}} p_{n+1} \mathbb{E} \xi^2(X_1 X_2 \cdots X_n X_{n+1}) \\
&\leq (2d+1) C \sum_{n=1}^{\infty} p_n \mathbb{E} \xi^2(X_1 X_2 \cdots X_n X_{n+1}) \\
&= (2d+1) C \|\xi\|^2.
\end{aligned}$$

Thus  $\|S_a\| \leq \sqrt{(2d+1)C}$  as required. Finally, the way the operators  $S_a$  are defined for  $a \in \mathfrak{a}$  ensures that the map  $g \mapsto S_g$  is unambiguously defined and is a representation of  $G$ .  $\square$

**Lemma 2.2.** *For each  $b \in G$  there is a positive constant  $M_b$  such that*

$$\frac{1}{M_b} w(g) \leq w(gb) \leq M_b w(g),$$

for every  $g \in G$ .

*Proof.* For  $a \in \mathfrak{a}$  and  $g_0 \in G$ , applying the inequality

$$\|S_a \xi\|^2 \leq (2d+1) C \|\xi\|^2,$$

obtained in the proof of the previous lemma, to the element  $\xi_{g_0} \in l_2(G, w)$ , which is defined by

$$\xi_{g_0}(g) = \begin{cases} 1 & g = g_0 \\ 0 & \text{otherwise,} \end{cases}$$

we get

$$w(g_0 a^{-1}) \leq (2d+1) C w(g_0).$$

Writing  $g = g_0 a^{-1}$  and  $M_a = (2d+1)C$  we get

$$w(g) \leq M_a w(ga).$$

By the symmetry of  $\mathfrak{a}$  we get

$$\frac{1}{M_a} w(g) \leq w(ga) \leq M_a w(g), \quad \forall g \in G.$$

Writing  $b$  as a word on  $\mathfrak{a}$  and iterating we get the required constant  $M_b$ .  $\square$

**2.2. The construction of the measure.** Consider an arbitrary ergodic free  $G$ -action on a standard Lebesgue probability space  $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$ . Our goal in this section is to construct a  $G$ -invariant probability measure on  $H$  which will provide a linear hypercyclic model for  $\mathbf{X}$ .

To construct this measure on  $H$  we will construct a mapping from  $X$  to  $H$  and for this we will need an ergodic theorem for certain Markov operators. Such theorems were first established by V. Oseledec in 1965 [7]. We will use a version from a paper of Jones-Rosenblatt-Tempelmann [5, Theorem 3.11].

Given a symmetric probability measure  $\eta$  on  $G$  (i.e.  $\eta(E^{-1}) = \eta(E)$  for every measurable  $E \subset G$ ) and a  $G$ -action  $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$ , define the operator  $A_\eta : L_p(X, \mu) \rightarrow L_p(X, \mu)$ ,  $1 \leq p \leq \infty$  by

$$(A_\eta f)(x) = \sum_{g \in G} f(T_g x) \eta(g).$$

We have

$$(A_\eta^n f)(x) = \sum_{g \in G} f(T_g x) \eta^{*n}(g).$$

We will say that  $A_\eta$  is a *random walk operator*. For  $1 \leq p \leq \infty$  let  $P_{\mathcal{I}}$  denote the projection from  $L_p(X, \mathcal{B}, \mu)$  onto the subspace  $\mathcal{I}$  of  $T$ -invariant functions. A probability measure  $\eta$  on a topological group  $G$  is *strictly aperiodic* if the support of  $\eta$ ,  $S(\eta)$ , is not contained in a coset of a proper closed normal subgroup of  $G$ . We say that the support of  $\eta$  *generates*  $G$  if the smallest closed subgroup containing  $S(\eta)$  is all of  $G$ . (In order to avoid confusion we change, in the following theorem, some of the notations of the original statement.)

**Theorem 2.3** (J-R-T). *Let  $\eta$  be a symmetric, strictly aperiodic, regular probability measure on  $G$ . Let  $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$  be measure preserving  $G$ -action on a probability space. Then  $A_\eta^n f$  converges in norm to  $P_{\mathcal{I}} f$  for all  $f \in L_p(X, \mathcal{B}, \mu)$ ,  $1 \leq p < \infty$ , and  $A_\eta^n f(x)$  converges for a.e.  $x$  if  $f \in L_p(X, \mathcal{B}, \mu)$ ,  $1 < p \leq \infty$ .*

We can now prove our theorem for a finitely generated  $G$ .

**Theorem 2.4.** *For every ergodic probability measure preserving free  $G$ -action  $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$ , there exists a  $\{S_g\}_{g \in G}$ -invariant probability measure  $\nu$  on  $H = l_2(G, w)$  which is positive on every nonempty open subset of  $H$  and such that the  $G$ -actions  $\mathbf{X}$  and  $(H, \text{Borel}(H), \nu, \{S_g\}_{g \in G})$  are isomorphic.*

*Proof.* 1. **The Markov operator associated with  $\rho$ :** With  $\rho$  as in (2.1), define the operator  $A_\rho : L_1(X, \mu) \rightarrow L_1(X, \mu)$  by

$$(A_\rho f)(x) = \sum_{g \in G} f(T_g x) \rho(g).$$

Then  $A_\rho$  is a random walk operator and we have

$$(A_\rho^n f)(x) = \sum_{g \in G} f(T_g x) \rho^{*n}(g).$$

**2. The factor map  $\phi_f$ :** Given  $f \in L_4(X, \mu)$ , the series

$$\sum_{g \in G} f^2(T_g x) w(g)$$

converges a.e. on  $X$ . In fact, by Theorem 2.3, we have

$$\begin{aligned} \sum_{g \in G} f^2(T_g x) w(g) &= \sum_{g \in G} f^2(T_g x) \sum_{n=1}^{\infty} p_n \rho^{*n}(g) \\ &= \sum_{n=1}^{\infty} p_n \sum_{g \in G} f^2(T_g x) \rho^{*n}(g) \\ &= \sum_{n=1}^{\infty} p_n (A_\rho^n f^2)(x) < \infty. \end{aligned}$$

Thus the map  $x \mapsto \{f(T_g x)\}_{g \in G} \in \mathbb{R}^G$  defines a map  $\phi_f : X \rightarrow H$ ,

$$\phi_f(x) = \{f(T_g x)\}_{g \in G}.$$

Note that  $\mu$ -a.e.  $f(x) = \phi_f(x)(e)$ . We denote the corresponding push forward measure by

$$\nu_f = (\phi_f)_*(\mu).$$

Moreover, we have

$$\phi_f(T_h x) = \{f(T_g T_h x)\}_{g \in G} = \{f(T_{gh} x)\}_{g \in G} = S_h \{f(T_g x)\}_{g \in G} = S_h \phi_f(x),$$

so that  $\phi_f : \mathbf{X} \rightarrow (H, \text{Borel}(H), \nu, \{S_g\}_{g \in G})$  is a factor map. Our proof will be complete when we show that there exists a function  $f \in L_4(X, \mu)$  for which (i) the map  $\phi_f$  is an isomorphism of dynamical systems and (ii) the support of the measure  $\nu_f$  is all of  $H$ ; i.e.  $\nu_f(U) > 0$  for every nonempty open subset of  $H$ .

**3. The induction scheme for the construction of  $f$ :** We first choose a sequence of measurable sets  $\{A_n\}$  which are dense in the measure algebra  $(\mathcal{B}, \mu)$ , and such that each element repeats infinitely often. Next we pick a sequence of balls  $\{U_n\}$  which form a basis for the topology of  $H = l_2(G, w)$ . More specifically,  $U_n = B_{r_n}(\xi_n)$  is the open ball of radius  $r_n > 0$  centered at  $\xi_n$ , where  $\xi_n \in H_{fin}$ , the dense subspace of  $H$  consisting of functions of finite support. We will construct inductively a sequence of finite valued functions  $\{f_n\}_{n=0}^\infty$ , with  $f_0 = 0$ , which will converge in  $L_2(X, \mu)$  to the desired function  $f$ . We will also define auxiliary sequences of positive reals  $\epsilon_n$ ,  $\gamma_n$  and  $\delta_n$  decreasing to 0.

We now suppose that by stage  $n$  we already have the following structure:

- 1<sub>n</sub> A function  $f_n$ , admitting finitely many values: to each  $i \leq n$  there are two finite subsets  $(V_{i,0}^{(n)}, V_{i,1}^{(n)})$  of  $\mathbb{R}$  which are at least  $\epsilon_i(1 + \frac{1}{n})$  apart, and the range of  $f_n = \cup_{i \leq n} \{V_{i,0}^{(n)} \cup V_{i,1}^{(n)}\}$ .
- 2<sub>n</sub> There is a number  $\beta_n$  so that for each  $i \leq n$  the sets  $(\hat{V}_{i,0}^{(n)}, \hat{V}_{i,1}^{(n)})$ , which are obtained by covering each point in the sets  $(V_{i,0}^{(n)}, V_{i,1}^{(n)})$  with intervals of length  $\beta_n$ , are  $\epsilon_i(1 + \frac{1}{n+1})$  apart, and so that for each  $i < n$  and  $j \in \{0, 1\}$ ,  $\hat{V}_{i,j}^{(n)} \subset \hat{V}_{i,j}^{(n-1)}$ .
- 3<sub>n</sub> For every  $i \leq n$  and for all  $x \in X$ ,  $x \in A_i$  implies  $f_n(x) \in V_{i,0}^{(n)}$  and  $x \in X \setminus A_i$  implies  $f_n(x) \in V_{i,1}^{(n)}$ , with the exception of sets of measure  $< \gamma_i(1 - \frac{1}{n})$ .
- 4<sub>n</sub> On a set  $E_i^{(n)}$  of measure  $\geq \delta_i(1 + \frac{1}{n})$ ,  $\phi_{f_n}$  takes values in  $U_i$  for  $i \leq n$ , with  $E_i^{(j+1)} \subset E_i^{(j)}$  for  $j \leq n-1$ .

$5_n$

$$\|f_n\|_4 = \left( \int_X |f_n|^4 d\mu \right)^{\frac{1}{4}} < 1.$$

**4. Step  $n + 1$ :**

Next find a  $\eta > 0$  so small that if we change  $f_n$  by less than  $\eta$  on a set of measure at least  $1 - \eta$  to obtain  $f_{n+1}$ , then we will still satisfy the conditions  $1_{n+1}$  to  $5_{n+1}$  for  $i \leq n$  with  $f_{n+1}$ . Thus  $\eta$  has to be less than  $\min\{\frac{1}{2n(n+1)}\epsilon_i, \frac{1}{2n(n+1)}\beta_i, \frac{1}{2n(n+1)}\delta_i, \frac{1}{2n(n+1)}\gamma_i\}$  for all  $i \leq n$ .

(a) *Hitting  $U_{n+1}$  with positive probability:* Recall that  $U_{n+1} = B_{r_{n+1}}(\xi_{n+1}) \subset H$  where  $\xi_{n+1} \in H_{fin}$  is supported on a ball, say  $B_{N_0}$  in  $G$  (this is the collection of elements in  $G$  which have a representation as a word on the alphabet  $\mathbf{a}$  of length  $\leq N_0$ ).

As  $G$  acts freely on  $X$  we can find  $N \gg N_0$  (we will determine soon how large  $N$  needs to be) and a set  $E$  of positive measure such that the sets  $\{gE\}_{g \in B_N}$  are disjoint and  $\mu(B_N E) < \frac{1}{2}\eta$  (this number determines the new  $\delta_{n+1}$ ). For a proof of this fact see e.g. [8, Lemma 3.1]. We erase  $f_n$  on  $B_N E$  and replace it there by a function which (i) matches the function  $\xi_{n+1}$  on  $B_{N_0} E$  and (ii) is zero on the set  $(B_N \setminus B_{N_0})E$ . This new function we denote by  $\hat{f}_n$ . We only have to choose  $N$  sufficiently large so that a.e. the contribution of  $\hat{f}_n^2(gx)$  with  $g \notin B_N$  to the  $l_2$ -norm of  $\phi_{f_n}$  in  $H = l_2(G, w)$  is  $< \frac{1}{2}r_{n+1}$  (recall that  $\hat{f}_n$  admits only finitely many values).

(b) *Distinguishing  $A_{n+1}$  from  $X \setminus A_{n+1}$ :* Let  $(\hat{V}_{i,0}^{(n)}, \hat{V}_{i,1}^{(n)})$  be closed neighborhoods of the sets  $(V_{i,0}^{(n)}, V_{i,1}^{(n)})$ , obtained by covering each point with an interval of length  $< \beta_n$ , which are  $\epsilon_i(1 + \frac{1}{n+1})$  apart. We consider a set of constancy, say  $D \subset X$ , of the function  $\hat{f}_n$ , where it takes the value  $u \in \mathbb{R}$ . We split  $u$  to two values  $u_0$  and  $u_1$ , staying within the interiors of the corresponding sets  $(\hat{V}_{i,0}^{(n)}, \hat{V}_{i,1}^{(n)})$ , to distinguish between  $A_{n+1}$  and  $X \setminus A_{n+1}$ . We do this for all the values of  $\hat{f}_n$  to define the new sets  $(V_{n+1,0}^{(n+1)}, V_{n+1,1}^{(n+1)})$  and a new  $\epsilon_{n+1}$  (all the points are distinct). This also defines the new sets  $(V_{i,0}^{(n+1)}, V_{i,1}^{(n+1)})$  for  $i \leq n$  and the function  $f_{n+1}$ . Finally we choose a number  $\beta_{n+1}$  so that for each  $i \leq n + 1$  the sets  $(\hat{V}_{i,0}^{(n+1)}, \hat{V}_{i,1}^{(n+1)})$ , which are obtained by covering each point in the sets  $(V_{i,0}^{(n)}, V_{i,1}^{(n)})$  with intervals of length  $\beta_{n+1}$ , are  $\epsilon_i(1 + \frac{1}{n+2})$  apart, and so that for each  $i < n + 1$  and  $j \in \{0, 1\}$ ,  $\hat{V}_{i,j}^{(n+1)} \subset \hat{V}_{i,j}^{(n)}$ .

**5. Conclusion:** The way the sequence  $f_n$  was constructed ensures the existence in  $L_2(X, \mu)$  of the limit function  $f := \lim_{n \rightarrow \infty} f_n$ . The essential range of the function  $f$  lies in the countable union of Cantor sets

$$E = \bigcup_{i=1}^{\infty} C_{i,0} \cup C_{i,1},$$

where

$$C_{i,0} = \bigcap_{n=i}^{\infty} \hat{V}_{i,0}^{(n)} \quad \text{and} \quad C_{i,1} = \bigcap_{n=i}^{\infty} \hat{V}_{i,1}^{(n)},$$

and

$$C_{i,0} \cap C_{i,0} = \emptyset.$$

It is now easy to check that the properties  $3_n$  imply that the map  $\phi_f$  is an isomorphism. E.g. we have, for each  $i$

$$\mu(f^{-1}(C_{i,0}) \triangle A_i) < \gamma_i,$$

and, as each  $A = A_i$  appears infinitely often, we conclude that, in fact

$$f^{-1}(C_{i,0}) = A \pmod{0}.$$

Similarly the properties  $4_n$  imply that  $\text{supp}(\nu_f) = H$ . This completes the proof of Theorem 2.4.  $\square$

### 3. THE GENERAL COUNTABLE GROUP CASE

Our goal in this section is to prove Theorem 1.2. We do this by applying Theorem 2.4 to an ambient finitely generated group which is provided by the following well known result of Higman, Neumann and Neumann [4]:

**Theorem 3.1.** *Any countable group  $G$  can be embedded in a group  $H$  generated by two elements. If the number of defining relations for  $G$  is  $n$ , the number of defining relations for  $H$  can be taken to be  $n$ .*

*Proof of Theorem 1.2.* Let  $G < H$  be a HNN extension of  $G$  with  $H = \langle a_1^{\pm 1}, a_2^{\pm 1} \rangle$ . Construct the symmetric probability measure  $\{w_H(h) : h \in H\}$  as in the proof of Theorem 2.4. By Lemma 2.2 there is, for every  $b \in H$ , a constant  $M_b > 0$  with

$$\frac{1}{M_b} w_H(h) \leq w_H(hb) \leq M_b w_H(h),$$

for all  $h \in H$ . We restrict  $w_H$  to  $G$  and renormalize to obtain a probability measure  $\rho$  on  $G$ . Thus, for every  $g \in G$ ,  $\rho(g) = \frac{1}{K} w_H(g)$ , where  $K = \sum_{k \in G} w_H(k)$ . We still have that for every  $b \in G$

$$(3.1) \quad \frac{1}{M_b} \rho(g) \leq \rho(gb) \leq M_b \rho(g), \quad \forall g \in G.$$

Next we use the symmetric probability measure  $\rho$  — which is of course no longer finitely supported; in fact, it is strictly positive everywhere on  $G$  — to create a new probability measure  $w = w_G$  on  $G$ , namely  $w(g) = \sum_{n=1}^{\infty} p_n \rho^{*n}(g)$  for every  $g \in G$ .

Now in the Hilbert space  $l_2(G, w)$ , by an argument similar to the one used in the proof of Lemma 2.1, and using (3.1), we have for each  $g_0 \in G$  and every  $\xi \in l_2(G, w)$ :

$$\begin{aligned} \|S_{g_0} \xi\|^2 &= \sum_{g \in G} (S_{g_0} \xi)^2(g) w(g) \\ &= \sum_{n=1}^{\infty} \sum_{g_1, g_2, \dots, g_n} p_n \xi^2(g_1 g_2 \cdots g_n g_0) \rho(g_1) \rho(g_2) \cdots \rho(g_n) \\ &\leq M_{g_0} \sum_{n=1}^{\infty} \sum_{g_1, g_2, \dots, g_n} p_n \xi^2(g_1 g_2 \cdots g_{n-1} (g_n g_0)) \rho(g_1) \rho(g_2) \cdots \rho(g_{n-1}) \rho(g_n g_0) \\ &= M_{g_0} \sum_{n=1}^{\infty} \sum_{g_1, g_2, \dots, g_n} p_n \xi^2(g_1 g_2 \cdots g_n) \rho(g_1) \rho(g_2) \cdots \rho(g_n). \end{aligned}$$



Thus  $\|S_{g_0}\| \leq M_{g_0}$  and the map  $g \mapsto S_g$  is the required representation of  $G$  on  $l_2(G, w)$ . The rest of the proof follows mutatis mutandis that of Theorem 2.4.  $\square$

#### 4. THE COMPACTLY GENERATED GROUP CASE

In this section we deal with topological groups which are not necessarily discrete. Our goal is to obtain an analogue of Theorem 2.4 for a topological group which is compactly generated, locally compact and second countable. We will concentrate on pointing out the technical results that are needed to carry out the proof that we gave in detail for countable groups. The first lemma concerns the group itself equipped with a Haar measure.

**Lemma 4.1.** *Let  $G$  be a locally compact group and  $K$  a compact symmetric neighborhood of the identity. Let  $\lambda$  be a fixed right-invariant Haar measure on  $G$ . For any compact symmetric set  $L \subset G$  such that  $K$  lies in the interior of  $L$  (for example  $K^2$ ) if  $m$  is Haar measure restricted to  $L$  (i.e.  $m = \mathbf{1}_L \lambda$ ) then the density of the measure  $m * m$  restricted to  $LK$  is bounded away from zero.*

*Proof.* Since the density of  $m * m$ ; i.e. the function  $\psi$  for which  $d(m * m) = \psi(g) d\lambda(g)$ , is continuous, and  $LK$  is compact it suffices to show that for any point of the form  $lk$  with  $l \in L$  and  $k \in K$  this density is positive. Now this density is given by

$$\psi(lk) = \int \mathbf{1}_L(lkh^{-1}) \mathbf{1}_L(h) d\lambda(h).$$

For all  $v$  in a small symmetric neighborhood of the identity  $V$  we have  $vk \in L$ , since  $K$  is in the interior of  $L$ . For  $h = vk$  of this form we have  $lkh^{-1} = lv^{-1}$  and  $lV$  must intersect the interior of  $L$ . Thus this intersection has positive Haar measure, from which it follows that the density must be positive:

$$\begin{aligned} \psi(lk) &= \int \mathbf{1}_L(lkh^{-1}) \mathbf{1}_L(h) d\lambda(h) \\ &\geq \int \mathbf{1}_L(lv^{-1}) \mathbf{1}_L(vk) d\lambda(vk) > 0. \end{aligned}$$

$\square$

We remark that a similar conclusion is valid also for  $\hat{m}$  which is defined by  $\hat{m}(B) = m(B^{-1})$ .

Let now  $K$  be a symmetric compact neighborhood of the identity which generates  $G$  (i.e.  $\cup_{n \in \mathbb{N}} K^n = G$ ). Let  $L = K^2$  and define  $\rho$  by the following equation:

$$\rho(B) = \frac{1}{2\lambda(L)} (\lambda(L \cap B) + \lambda(L \cap B^{-1})).$$

By definition  $\rho = \frac{1}{2}(m + \hat{m})$  is a symmetric measure so that when it averages unitary operators the result will be self adjoint.

Finally let

$$w = \sum_{n=1}^{\infty} p_n \rho^{*n},$$

where, as in Section 2,  $p_1, p_2, \dots$  is a sequence of positive real numbers with  $\sum_{n=1}^{\infty} p_n = 1$  and such that for some  $C > 0$ ,  $\frac{p_n}{p_{n+1}} \leq C$  for every  $n$ .

Our Hilbert space now is  $H = L_2(G, w)$  and the representation is given by right multiplication:  $S_k \xi(g) = \xi(gk)$  for  $k \in K$ .

We can now formulate the main result of this section:

**Theorem 4.2.** *Let  $G$  be a topological group which is compactly generated, locally compact and second countable. For every ergodic probability measure preserving free  $G$ -action  $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$ , there exists a  $\{S_g\}_{g \in G}$ -invariant probability measure  $\nu$  on  $H = L_2(G, w)$  which is positive on every nonempty open subset of  $H$  and such that the  $G$ -actions  $\mathbf{X}$  and  $(H, \text{Borel}(H), \nu, \{S_g\}_{g \in G})$  are isomorphic.*

The fact that the shift operators  $S_g$  are bounded is shown as follows. From Lemma 4.1 we deduce:

**Lemma 4.3.** *There exists a constant  $D > 0$  such that for every  $k \in K$ ,  $g \in G$  and  $n \in \mathbb{N}$ :*

1.  $\rho * \delta_k \leq D(\rho * \rho)$ , hence also

$$\rho^{*n} * \delta_k \leq D\rho^{*(n+1)}, \quad \forall n \geq 1.$$

2.  $dw(gk) \leq DC dw(g)$ .

*Proof.* 1. Applying Lemma 4.1 we know that  $u := \min\{\psi(t) : t \in KL\} > 0$ . We then have on  $Lk$ ,  $m * m \geq u m * \delta_k$ . In a similar fashion we will have a positive constant  $v$  such that  $\hat{m} * \hat{m} \geq v \hat{m} * \delta_k$ . Thus we obtain  $\rho * \delta_k \leq \frac{2}{u \wedge v} \rho * \rho$ . Convolving with  $\rho^{*n}$  on the left we finally get

$$\rho^{*n} * \delta_k \leq D\rho^{*(n+1)},$$

with  $D = \frac{2}{u \wedge v}$ .

2. As in the proof of Lemma 2.1. □

**Corollary 4.4.** *For every  $k \in K$  we have:*

$$\|S_k\| \leq \sqrt{DC}.$$

For the main part of the proof of the theorem we will need a continuous version of the basic property of free actions of countable groups that we used in the previous section. Since this is probably less familiar we will spell this out in detail.

**Definition 4.5.** Given a  $G$  probability measure preserving action  $\mathbf{X} = (X, \mathcal{B}, \mu, \{T_g\}_{g \in G})$ , and a compact neighborhood  $Q$  of  $e \in G$ , we say that a measurable subset  $V \subset X$  is a *thin base for a  $Q$ -tower*, if there is a Borel measure  $\theta$  on  $V$  such that the action map  $\alpha : Q \times V \rightarrow \alpha(Q \times V) \subset X$ , given by  $\alpha(g, v) = T_g v$ , is one-to-one and maps  $\mathbf{1}_Q \lambda \times \theta$  to  $\mathbf{1}_{\alpha(Q \times V)} \mu$ .

The following property holds for free actions of locally compact groups. If  $Q$  is an arbitrary compact symmetric neighborhood of  $e \in G$  and  $E \subset X$  is any measurable subset with  $\mu(E) > 0$  then, as the action is free, by [6, page 54],  $E$  contains a subset  $V \subset E$  which is a thin base for a  $Q$ -tower. The construction of the function  $f$  on  $X$  will be carried out by defining a sequence of finite valued functions  $f_n$  on  $Q_n$ -towers as in the proof of Theorem 2.4. The function space  $L_2(G, w)$  contains a countable set of finite valued functions that is dense and we can use them to define a countable collection of open balls  $U_n$  as before. There is an added technical complication which arises from the fact that the thin bases of the  $Q$ -towers have measure zero. This is

taken care of by the basic fact that in  $L_2(G, w)$  the translation by elements of  $G$  is strongly continuous, so that having defined a function using a thin tower for a set of positive measure we will have functions which are very close to it.

Having said all of this one can basically copy the proof that we spelled out in detail and establish the theorem.

$$5. \quad G = \cup K_n$$

In this short section we will indicate how one obtains a proof of a version of Theorem 4.2 for a locally compact, second countable group  $G$  which is not compactly generated but has a rather simple structure:  $G = \cup_{n=1}^{\infty} K_n$ , where  $K_1 < K_2 < \dots$  is an increasing sequence of compact open subgroups.

**Theorem 5.1.** *The statement of Theorem 4.2 holds for every topological group  $G$  as above.*

*Proof.* As is the case of Theorem 4.2, we only need to demonstrate an analogue of the key inequality of Lemma 4.3. The role of  $\rho$  is now played by the probability measure

$$\rho = \sum_{n=1}^{\infty} p_n \lambda_n,$$

where  $\lambda_n$  is the normalized Haar measure of the compact subgroup  $K_n$ .

**Lemma 5.2.** *For every  $g_0 \in G$  there exists a constant  $C_{g_0}$  such that*

$$\rho * \delta_{g_0} \leq C_{g_0} (\rho * \rho).$$

*Proof.* Let  $m_0$  be the first integer with  $g_0 \in K_{m_0}$ . Then

$$\rho * \delta_{g_0} = \sum_{n=1}^{\infty} p_n \lambda_n * \delta_{g_0} = \sum_{n=1}^{m_0-1} p_n \lambda_n * \delta_{g_0} + \sum_{n=m_0}^{\infty} p_n \lambda_n.$$

Moreover, for every  $n \leq m_0$  we have the inequality

$$\lambda_n * \delta_{g_0} \leq [K_{m_0} : K_n] \lambda_{m_0} \leq [K_{m_0} : K_1] \lambda_{m_0}.$$

On the other hand an easy computation shows that

$$\begin{aligned} \rho * \rho &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i p_j \lambda_i * \lambda_j \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i p_j \lambda_{i \vee j} \\ &\geq \sum_{k=1}^{\infty} \left( \sum_{i \leq k} p_i \right) p_k \lambda_k \\ &\geq p_1 \sum_{k=1}^{\infty} p_k \lambda_k. \end{aligned}$$

Thus taking  $C_{g_0} = \frac{1}{p_1} + [K_{m_0} : K_1]$  we obtain the required inequality. □

Again we finish the proof as in Theorem 4.2. □

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